MAE 656 - Advanced Computer Aided Design

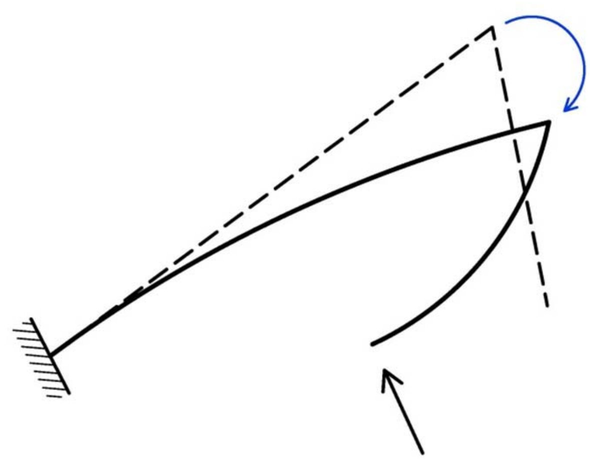
01. Introduction – Doc 04

Numerical Simulation of Beam Elements

Introduction

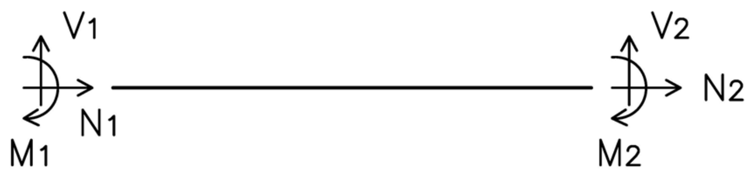
Most of the exiting bar structures have rigid connections between the bars instead of hinges.

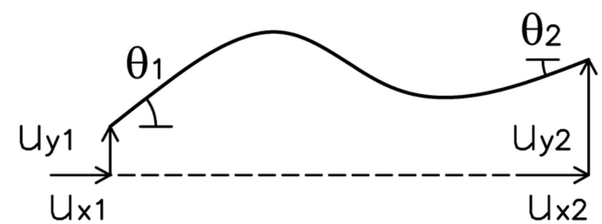
A rigid connection implies that all movements (displacements and rotations) are transferred from one bar to the other one.



Formulation

Now, for a single bar, we have to define the relation between three efforts and three movements:





$$\begin{bmatrix} \overrightarrow{f_1} \\ \overrightarrow{f_2} \end{bmatrix} = \begin{bmatrix} k_{11} & k_{12} \\ k_{21} & k_{22} \end{bmatrix} \cdot \begin{bmatrix} \overrightarrow{a_1} \\ \overrightarrow{a_2} \end{bmatrix}$$

with, $\overrightarrow{f_i} = \begin{bmatrix} n_i \\ v_i \\ m_i \end{bmatrix}$ and $\overrightarrow{a_i} = \begin{bmatrix} u_{xi} \\ u_{yi} \\ \theta_i \end{bmatrix}$

Formulation

And each k_{ij} matrix has the following expression:

$$k_{11} = \begin{bmatrix} +\frac{EA}{L} & 0 & 0 \\ 0 & +\frac{12EI}{L^3} & +\frac{6EI}{L^2} \\ 0 & +\frac{6EI}{L^2} & +\frac{4EI}{L} \end{bmatrix} \qquad k_{12} = \begin{bmatrix} -\frac{EA}{L} & 0 & 0 \\ 0 & -\frac{12EI}{L^3} & +\frac{6EI}{L^2} \\ 0 & -\frac{6EI}{L^2} & +\frac{2EI}{L} \end{bmatrix}$$

$$k_{21} = \begin{bmatrix} -\frac{EA}{L} & 0 & 0 \\ 0 & -\frac{12EI}{L^3} & -\frac{6EI}{L^2} \\ 0 & +\frac{6EI}{L^2} & +\frac{2EI}{L} \end{bmatrix} \qquad k_{22} = \begin{bmatrix} +\frac{EA}{L} & 0 & 0 \\ 0 & +\frac{12EI}{L^3} & -\frac{6EI}{L^2} \\ 0 & -\frac{6EI}{L^2} & +\frac{4EI}{L} \end{bmatrix}$$

Formulation

Writing everything together we get

$$\begin{bmatrix} n_1 \\ v_1 \\ m_1 \\ n_2 \\ v_2 \\ m_2 \end{bmatrix} = \begin{bmatrix} +\frac{EA}{L} & 0 & 0 & -\frac{EA}{L} & 0 & 0 \\ 0 & +\frac{12EI}{L^3} & +\frac{6EI}{L^2} & 0 & -\frac{12EI}{L^3} & +\frac{6EI}{L^2} \\ 0 & +\frac{6EI}{L^2} & +\frac{4EI}{L} & 0 & -\frac{6EI}{L^2} & +\frac{2EI}{L} \\ -\frac{EA}{L} & 0 & 0 & +\frac{EA}{L} & 0 & 0 \\ 0 & -\frac{12EI}{L^3} & -\frac{6EI}{L^2} & 0 & +\frac{12EI}{L^3} & -\frac{6EI}{L^2} \\ 0 & +\frac{6EI}{L^2} & +\frac{2EI}{L} & 0 & -\frac{6EI}{L^2} & +\frac{4EI}{L} \end{bmatrix} \cdot \begin{bmatrix} u_{x1} \\ u_{y1} \\ \theta_1 \\ u_{x2} \\ u_{y2} \\ \theta_2 \end{bmatrix}$$

Formulation

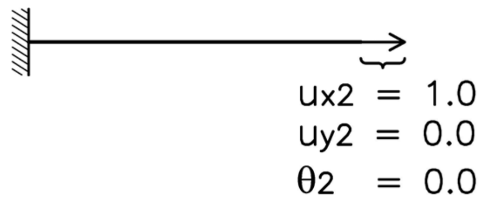
Where do these values come from?

Let's study how is affected the force array if we modify one displacement:

$$\begin{bmatrix} n_1 \\ v_1 \\ m_1 \\ n_2 \\ v_2 \\ m_2 \end{bmatrix} = \begin{bmatrix} +\frac{EA}{L} & 0 & 0 & -\frac{EA}{L} & 0 & 0 \\ 0 & +\frac{12EI}{L^3} & +\frac{6EI}{L^2} & 0 & -\frac{12EI}{L^3} & +\frac{6EI}{L^2} \\ 0 & +\frac{6EI}{L^2} & +\frac{4EI}{L} & 0 & -\frac{6EI}{L^2} & +\frac{2EI}{L} \\ -\frac{EA}{L} & 0 & 0 & +\frac{EA}{L} & 0 & 0 \\ 0 & -\frac{12EI}{L^3} & -\frac{6EI}{L^2} & 0 & +\frac{12EI}{L^3} & -\frac{6EI}{L^2} \\ 0 & +\frac{6EI}{L^2} & +\frac{2EI}{L} & 0 & -\frac{6EI}{L^2} & +\frac{4EI}{L} \end{bmatrix} \cdot \begin{bmatrix} u_{x1} \\ u_{y1} \\ \theta_1 \\ u_{x2} \\ u_{y2} \\ \theta_2 \end{bmatrix}$$

Formulation

This is, if we apply a displacement:



We have to apply a force in node 2:

$$N_2 = \frac{E \cdot A}{L} \cdot u_{x2} = \frac{E \cdot A}{L} \cdot 1$$

And we will obtain a reaction force in node 1:

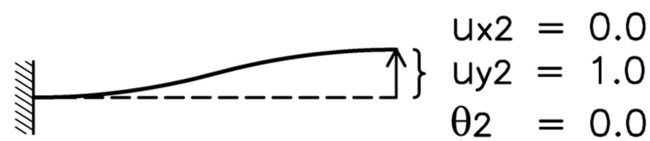
$$N_1 = -\frac{E \cdot A}{L} \cdot 1$$

This is what it is written in the stiffness matrix!

Formulation

Let's study another case:

$$
\begin{bmatrix} n_1 \\ v_1 \\ m_1 \\ n_2 \\ v_2 \\ m_2 \end{bmatrix} =
\begin{bmatrix}
+\frac{EA}{L} & 0 & 0 & -\frac{EA}{L} & 0 & 0 \\
0 & +\frac{12EI}{L^3} & +\frac{6EI}{L^2} & 0 & -\frac{12EI}{L^3} & +\frac{6EI}{L^2} \\
0 & +\frac{6EI}{L^2} & +\frac{4EI}{L} & 0 & -\frac{6EI}{L^2} & +\frac{2EI}{L} \\
-\frac{EA}{L} & 0 & 0 & +\frac{EA}{L} & 0 & 0 \\
0 & -\frac{12EI}{L^3} & -\frac{6EI}{L^2} & 0 & +\frac{12EI}{L^3} & -\frac{6EI}{L^2} \\
0 & +\frac{6EI}{L^2} & +\frac{2EI}{L} & 0 & -\frac{6EI}{L^2} & +\frac{4EI}{L}
\end{bmatrix} \cdot
\begin{bmatrix} u_{x1} \\ u_{y1} \\ \theta_1 \\ u_{x2} \\ u_{y2} \\ \theta_2 \end{bmatrix}
$$



We have to apply in node 2:

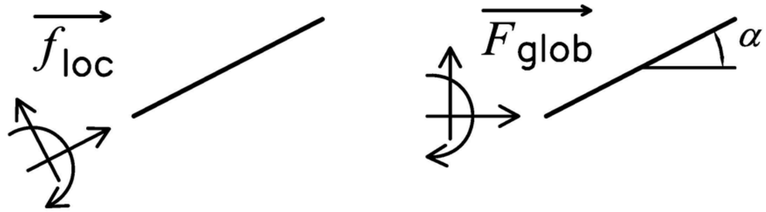
$$
\begin{cases} V_2 = \dfrac{12EI}{L^3} \cdot 1 \\[2ex] M_2 = -\dfrac{6EI}{L^2} \cdot 1 \end{cases}
$$

And we will obtain a reaction force in node 1:

$$
\begin{cases} V_1 = -\dfrac{12EI}{L^3} \cdot 1 \\[2ex] M_1 = -\dfrac{6EI}{L^2} \cdot 1 \end{cases}
$$

Formulation

If the bar is not horizontal, we will have to rotate it:



Defining L as:

$$L = \begin{bmatrix} \cos\alpha & \sin\alpha & 0 \\ -\sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

The relation between global and local is:

$\overrightarrow{f_{loc}} = L \cdot \overrightarrow{f_{glob}}$ and $\overrightarrow{a_{loc}} = L \cdot \overrightarrow{a_{glob}}$

$\overrightarrow{f_{glob}} = L^T \cdot \overrightarrow{f_{loc}}$ and $\overrightarrow{a_{glob}} = L^T \cdot \overrightarrow{a_{loc}}$

Formulation

Knowing the relation between local and global for any force or vector, the rotation matrix is defined as:

$$T = \begin{bmatrix} L & 0 \\ 0 & L \end{bmatrix}$$

The rotation matrix is applied to the force and displacement vector of the beam. Using the known relation between these two fields, it is possible to obtain the expression of the stiffness matrix in global coordinates:

$$\overrightarrow{f_{loc}} = k_{loc} \cdot \overrightarrow{a_{loc}}$$

$$\overrightarrow{f_{glob}} = T^T \cdot \overrightarrow{f_{loc}} = T^T \cdot k_{loc} \cdot \overrightarrow{a_{loc}} = T^T \cdot k_{loc} \cdot T \cdot \overrightarrow{a_{glob}}$$

$$k_{glob} = T^T \cdot k_{loc} \cdot T$$

Implementation

Once knowing the expression of the stiffness matrix, it is possible to calculate it for the different bar elements that compose the structure.

Once the stiffness matrix of each beam element is calculated for each element, it has to be assembled in the global stiffness matrix of the structure, based on the bar connectivities:

$$\begin{bmatrix} \vdots \\ \vec{f_i} \\ \vdots \\ \vec{f_j} \end{bmatrix} = \begin{bmatrix} \cdots & \vdots & \cdots & \vdots \\ \cdots & {k_{11}}^e & \cdots & {k_{12}}^e \\ & \vdots & & \vdots \\ \cdots & {k_{21}}^e & \cdots & {k_{22}}^e \end{bmatrix} \begin{bmatrix} \vdots \\ \vec{a_i} \\ \vdots \\ \vec{a_j} \end{bmatrix}$$

→ Row i

→ Row j

→ Column j

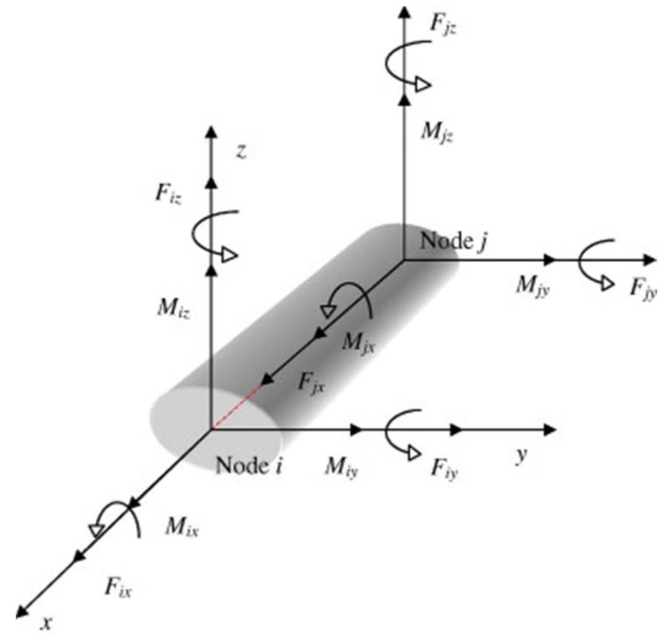
→ Column i

Now, each k_{ij} is a 3x3 matrix!

3D Case

In a 3D case, instead of three forces in each node (axial, shear, and bending moment) we will have 6 forces.

And, instead of 3 movements we will also have 6.



Therefore, the stiffness matrix of the structure will be of size 12x12 (divided in 4 k_{ij} matrices of 6x6)

3D Case

$$
\begin{bmatrix} f_{x1} \\ f_{y1} \\ f_{z1} \\ m_{x1} \\ m_{y1} \\ m_{z1} \\ f_{x2} \\ f_{y2} \\ f_{z2} \\ m_{x2} \\ m_{y2} \\ m_{z2} \end{bmatrix}
\quad
\underline{\hat{k}} = \left[\begin{array}{cccccc|cccccc}
\frac{AE}{L} & 0 & 0 & 0 & 0 & 0 & -\frac{AE}{L} & 0 & 0 & 0 & 0 & 0 \\
0 & \frac{12EI_z}{L^3} & 0 & 0 & 0 & \frac{6EI_z}{L^2} & 0 & -\frac{12EI_z}{L^3} & 0 & 0 & 0 & \frac{6EI_z}{L^2} \\
0 & 0 & \frac{12EI_y}{L^3} & 0 & -\frac{6EI_y}{L^2} & 0 & 0 & 0 & -\frac{12EI_y}{L^3} & 0 & -\frac{6EI_y}{L^2} & 0 \\
0 & 0 & 0 & \frac{GJ}{L} & 0 & 0 & 0 & 0 & 0 & -\frac{GJ}{L} & 0 & 0 \\
0 & 0 & -\frac{6EI_y}{L^2} & 0 & \frac{4EI_y}{L} & 0 & 0 & 0 & \frac{6EI_y}{L^2} & 0 & \frac{2EI_y}{L} & 0 \\
0 & \frac{6EI_z}{L^2} & 0 & 0 & 0 & \frac{4EI_z}{L} & 0 & -\frac{6EI_z}{L^2} & 0 & 0 & 0 & \frac{2EI_z}{L} \\
\hline
-\frac{AE}{L} & 0 & 0 & 0 & 0 & 0 & \frac{AE}{L} & 0 & 0 & 0 & 0 & 0 \\
0 & -\frac{12EI_z}{L^3} & 0 & 0 & 0 & -\frac{6EI_z}{L^2} & 0 & \frac{12EI_z}{L^3} & 0 & 0 & 0 & -\frac{6EI_z}{L^2} \\
0 & 0 & -\frac{12EI_y}{L^3} & 0 & \frac{6EI_y}{L^2} & 0 & 0 & 0 & \frac{12EI_y}{L^3} & 0 & \frac{6EI_y}{L^2} & 0 \\
0 & 0 & 0 & -\frac{GJ}{L} & 0 & 0 & 0 & 0 & 0 & \frac{GJ}{L} & 0 & 0 \\
0 & 0 & -\frac{6EI_y}{L^2} & 0 & \frac{2EI_y}{L} & 0 & 0 & 0 & \frac{6EI_y}{L^2} & 0 & \frac{4EI_y}{L} & 0 \\
0 & \frac{6EI_z}{L^2} & 0 & 0 & 0 & \frac{2EI_z}{L} & 0 & -\frac{6EI_z}{L^2} & 0 & 0 & 0 & \frac{4EI_z}{L}
\end{array}\right]
\begin{bmatrix} u_{x1} \\ u_{y1} \\ u_{z1} \\ \theta_{x1} \\ \theta_{y1} \\ \theta_{z1} \\ u_{x2} \\ u_{y2} \\ u_{z2} \\ \theta_{x2} \\ \theta_{y2} \\ \theta_{z2} \end{bmatrix}
$$

3D Case

Rotation of the stiffness matrix:

$$k_{glob} = T^T \cdot k_{loc} \cdot T$$

with:

$$T = \begin{bmatrix} L & & & \\ & L & & \\ & & L & \\ & & & L \end{bmatrix}$$

and:

$$L = \begin{bmatrix} \cos\theta_{Xx} & \cos\theta_{Yx} & \cos\theta_{Zx} \\ \cos\theta_{Xy} & \cos\theta_{Yy} & \cos\theta_{Zy} \\ \cos\theta_{Xz} & \cos\theta_{Yz} & \cos\theta_{Zz} \end{bmatrix}$$

Being X, Y, Z the global axes and x, y, z the local ones