MAE 656 - Advanced Computer Aided Design

01. Introduction – Doc 03

Numerical Simulation of Truss Elements

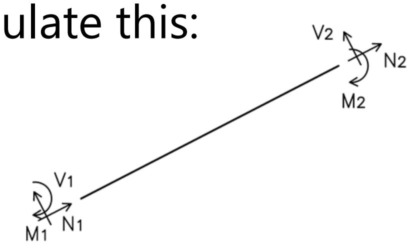
Introduction

The main aim of this lesson is to present the basic of a numerical calculation:

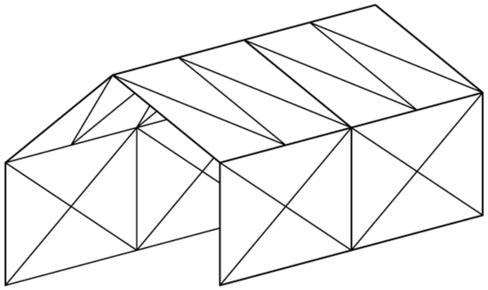
If we can transform a calculation in a repetitive procedure we can send it to a computer and it will do it for us.

A computer cannot think but can perform calculations substantially faster than us.

If we tell the computer how to calculate this:



It can afterwards calculate this:



Or this:

Introduction

To understand the process, we will start with the resolution of a very simple problem:

A straight truss bar (between hinges)

Afterwards we will extend the problem to

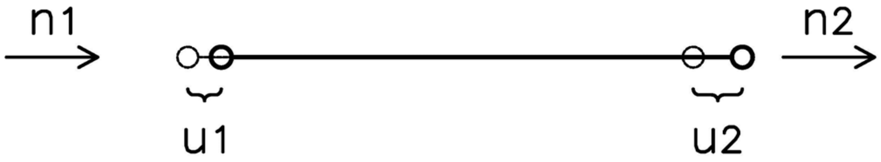
2D truss elements and 2D beam elements

And we will finish with the simulation of

2D solid elements with the Finite Element Method

This procedure will show us that the element required by any type of simulation (from the most simple to the most complicated) require always the definition of the same parameters!

Single bar between hinges



$$\varepsilon = \frac{u_2 - u_1}{L}$$

$$\sigma = E \cdot \varepsilon = E \cdot \frac{u_2 - u_1}{L}$$

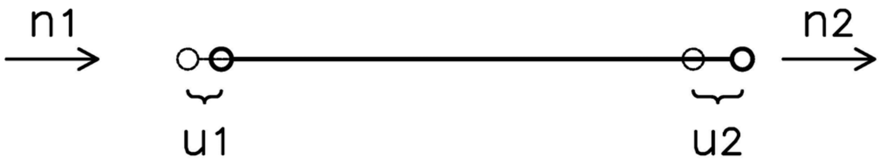
$$N = \sigma \cdot A = \frac{E \cdot A}{L} \cdot (u_2 - u_1)$$

if the bar is in equilibrium:

$$\sum F = 0; \quad n_1 = n_2$$

$$n_2 = N \quad \rightarrow \quad n_1 = -N$$

Single bar between hinges



The forces of each node can be computed as:

$$\begin{cases} n_1 = -\dfrac{E\cdot A}{L}\cdot(u_2-u_1) \\ n_2 = +\dfrac{E\cdot A}{L}\cdot(u_2-u_1) \end{cases}$$

Which can be written as a matrix:

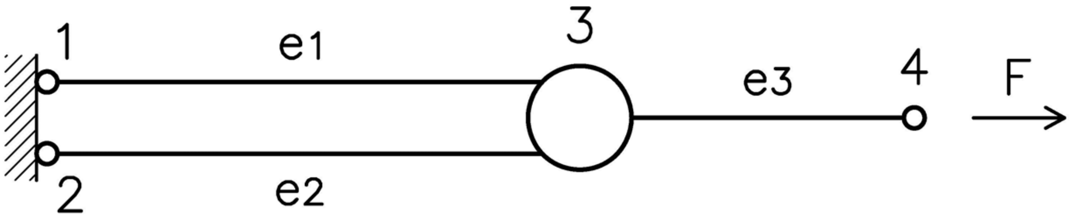
$$\begin{bmatrix} n_1 \\ n_2 \end{bmatrix} = \begin{bmatrix} +\dfrac{E\cdot A}{L} & -\dfrac{E\cdot A}{L} \\ -\dfrac{E\cdot A}{L} & +\dfrac{E\cdot A}{L} \end{bmatrix}\cdot\begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$$

$$\begin{bmatrix} n_1 \\ n_2 \end{bmatrix} = \overbrace{\frac{E\cdot A}{L}\cdot\begin{bmatrix} +1 & -1 \\ -1 & +1 \end{bmatrix}}^{\text{element stiffness matrix}}\cdot\begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$$

Thus, it is possible to obtain the forces on each node of a truss from the displacement of its nodes:

Three bars between hinges

Let’s see how the previous formulation can be used to solve:



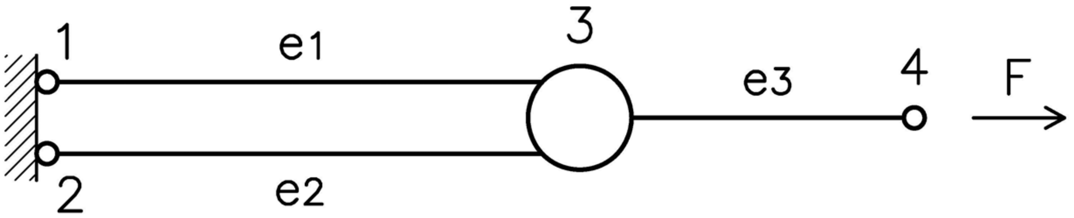
For each bar we can write:

$$\begin{bmatrix} {n_1}^{e1} \\ {n_2}^{e1} \end{bmatrix} = \frac{E^{e1} \cdot A^{e1}}{L^{e1}} \cdot \begin{bmatrix} +1 & -1 \\ -1 & +1 \end{bmatrix} \cdot \begin{bmatrix} {u_1}^{e1} \\ {u_2}^{e1} \end{bmatrix}$$

$$\begin{bmatrix} {n_1}^{e2} \\ {n_2}^{e2} \end{bmatrix} = \frac{E^{e2} \cdot A^{e2}}{L^{e2}} \cdot \begin{bmatrix} +1 & -1 \\ -1 & +1 \end{bmatrix} \cdot \begin{bmatrix} {u_1}^{e2} \\ {u_2}^{e2} \end{bmatrix}$$

$$\begin{bmatrix} {n_1}^{e3} \\ {n_2}^{e3} \end{bmatrix} = \frac{E^{e3} \cdot A^{e3}}{L^{e3}} \cdot \begin{bmatrix} +1 & -1 \\ -1 & +1 \end{bmatrix} \cdot \begin{bmatrix} {u_1}^{e3} \\ {u_2}^{e3} \end{bmatrix}$$

Three bars between hinges



Previous three equations are valid for any case. However, for our particular structure:

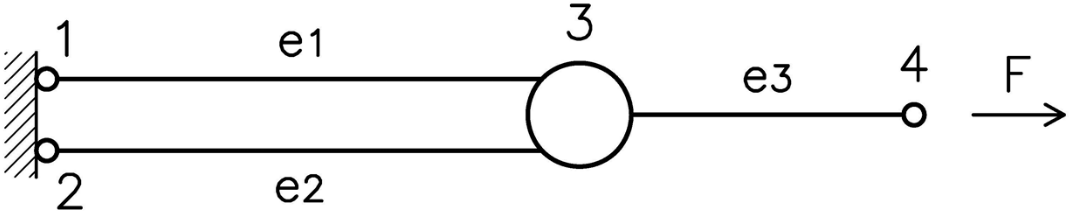
$$u_1^{e1} = U_1; \qquad u_2^{e1} = U_3 \qquad\qquad n_1^{e1} = N_1; \qquad n_2^{e1} = N_3^{e1}$$

$$u_1^{e2} = U_2; \qquad u_2^{e2} = U_3 \qquad\qquad n_1^{e2} = N_2; \qquad n_2^{e2} = N_3^{e2}$$

$$u_1^{e3} = U_3; \qquad u_2^{e3} = U_4 \qquad\qquad n_1^{e3} = N_3^{e3}; \qquad n_2^{e3} = N_4$$

and $N_3 = N_3^{e1} + N_3^{e2} + N_3^{e3}$

Three bars between hinges



Therefore:

$$\begin{bmatrix} N_1 \\ {N_3}^{e1} \end{bmatrix} = k_1 \cdot \begin{bmatrix} +1 & -1 \\ -1 & +1 \end{bmatrix} \cdot \begin{bmatrix} U_1 \\ U_3 \end{bmatrix}$$

$$\begin{bmatrix} N_2 \\ {N_3}^{e2} \end{bmatrix} = k_2 \cdot \begin{bmatrix} +1 & -1 \\ -1 & +1 \end{bmatrix} \cdot \begin{bmatrix} U_2 \\ U_3 \end{bmatrix}$$

$$\begin{bmatrix} {N_3}^{e3} \\ N_4 \end{bmatrix} = k_3 \cdot \begin{bmatrix} +1 & -1 \\ -1 & +1 \end{bmatrix} \cdot \begin{bmatrix} U_3 \\ U_4 \end{bmatrix}$$

with, $k_i = \frac{E^{ei} \cdot A^{ei}}{L^{ei}}$

We've been able to relate all forces and displacements of the structure using the stiffness of all bars composing it!

Three bars between hinges

having:

$$\begin{bmatrix} N_1 \\ {N_3}^{e1} \end{bmatrix} = k_1 \cdot \begin{bmatrix} +1 & -1 \\ -1 & +1 \end{bmatrix} \cdot \begin{bmatrix} U_1 \\ U_3 \end{bmatrix} \qquad \begin{bmatrix} N_2 \\ {N_3}^{e2} \end{bmatrix} = k_2 \cdot \begin{bmatrix} +1 & -1 \\ -1 & +1 \end{bmatrix} \cdot \begin{bmatrix} U_2 \\ U_3 \end{bmatrix}$$

$$\begin{bmatrix} {N_3}^{e3} \\ N_4 \end{bmatrix} = k_3 \cdot \begin{bmatrix} +1 & -1 \\ -1 & +1 \end{bmatrix} \cdot \begin{bmatrix} U_3 \\ U_4 \end{bmatrix}$$

we can write now the global system of equations:

$$\begin{bmatrix} N_1 \\ N_2 \\ N_3 \\ N_4 \end{bmatrix} = \underbrace{\begin{bmatrix} +k_1 & & -k_1 & \\ & +k_2 & -k_2 & \\ -k_1 & -k_2 & +k_1 + k_2 + k_3 & -k_3 \\ & & -k_3 & +k_3 \end{bmatrix}}_{\text{structure stiffness matrix}} \begin{bmatrix} U_1 \\ U_2 \\ U_3 \\ U_4 \end{bmatrix}$$

Three bars between hinges

having:

$$\begin{bmatrix} N_1 \\ {N_3}^{e1} \end{bmatrix} = k_1 \cdot \begin{bmatrix} +1 & -1 \\ -1 & +1 \end{bmatrix} \cdot \begin{bmatrix} U_1 \\ U_3 \end{bmatrix} \qquad \begin{bmatrix} N_2 \\ {N_3}^{e2} \end{bmatrix} = k_2 \cdot \begin{bmatrix} +1 & -1 \\ -1 & +1 \end{bmatrix} \cdot \begin{bmatrix} U_2 \\ U_3 \end{bmatrix}$$

$$\begin{bmatrix} {N_3}^{e3} \\ N_4 \end{bmatrix} = k_3 \cdot \begin{bmatrix} +1 & -1 \\ -1 & +1 \end{bmatrix} \cdot \begin{bmatrix} U_3 \\ U_4 \end{bmatrix}$$

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Three bars between hinges

having:

$$\begin{bmatrix} N_1 \\ {N_3}^{e1} \end{bmatrix} = k_1 \cdot \begin{bmatrix} +1 & -1 \\ -1 & +1 \end{bmatrix} \cdot \begin{bmatrix} U_1 \\ U_3 \end{bmatrix} \qquad \begin{bmatrix} N_2 \\ {N_3}^{e2} \end{bmatrix} = k_2 \cdot \begin{bmatrix} +1 & -1 \\ -1 & +1 \end{bmatrix} \cdot \begin{bmatrix} U_2 \\ U_3 \end{bmatrix}$$

$$\begin{bmatrix} {N_3}^{e3} \\ N_4 \end{bmatrix} = k_3 \cdot \begin{bmatrix} +1 & -1 \\ -1 & +1 \end{bmatrix} \cdot \begin{bmatrix} U_3 \\ U_4 \end{bmatrix}$$

we can write now the global system of equations:

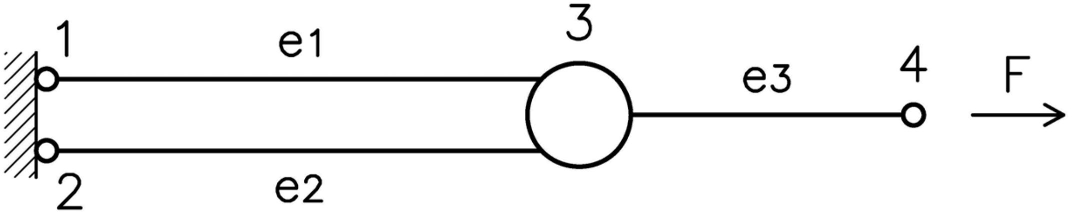
$$\begin{bmatrix} N_1 \\ N_2 \\ N_3 \\ N_4 \end{bmatrix} = \underbrace{\begin{bmatrix} +k_1 & & -k_1 & \\ & +k_2 & -k_2 & \\ -k_1 & -k_2 & +k_1 + k_2 + k_3 & -k_3 \\ & & -k_3 & +k_3 \end{bmatrix}}_{\text{structure stiffness matrix}} \begin{bmatrix} U_1 \\ U_2 \\ U_3 \\ U_4 \end{bmatrix}$$

Three bars between hinges

Can we solve this system?

$$\begin{bmatrix} N_1 \\ N_2 \\ N_3 \\ N_4 \end{bmatrix} = \begin{bmatrix} +k_1 & & -k_1 & \\ & +k_2 & -k_2 & \\ -k_1 & -k_2 & +k_1 + k_2 + k_3 & -k_3 \\ & & -k_3 & +k_3 \end{bmatrix} \begin{bmatrix} U_1 \\ U_2 \\ U_3 \\ U_4 \end{bmatrix}$$

What do we know?



$$U_1 = 0; \quad U_2 = 0; \quad U_3 = U_3; \quad U_4 = U_4$$

$$N_1 = N_1; \quad N_2 = N_2; \quad N_3 = 0; \quad N_4 = F$$

Three bars between hinges

therefore,

$$\begin{bmatrix} \cancel{N_1} \\ \cancel{N_2} \\ 0 \\ F \end{bmatrix} = \begin{bmatrix} \cancel{+k_1} & & \cancel{-k_1} & \\ & \cancel{+k_2} & \cancel{-k_2} & \\ \cancel{-k_1} & \cancel{-k_2} & +k_1 + k_2 + k_3 & -k_3 \\ & & -k_3 & +k_3 \end{bmatrix} \begin{bmatrix} \cancel{0} \\ \cancel{0} \\ U_3 \\ U_4 \end{bmatrix}$$

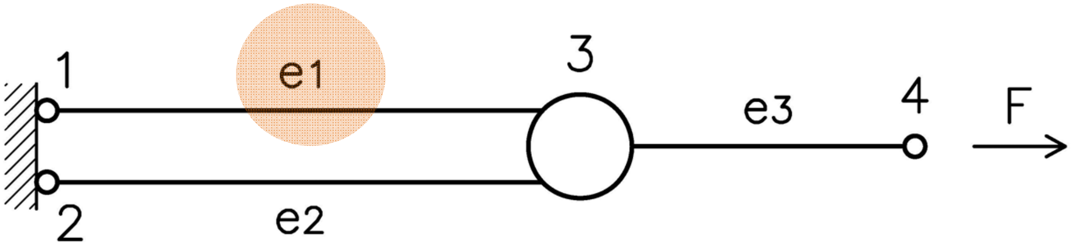
and the resulting system that has to be solved is:

$$\begin{bmatrix} 0 \\ F \end{bmatrix} = \begin{bmatrix} +k_1 + k_2 + k_3 & -k_3 \\ -k_3 & +k_3 \end{bmatrix} \begin{bmatrix} U_3 \\ U_4 \end{bmatrix}$$

Once knowing the values of U_3 and U_4, we can use the complete system to calculate reactions N_1 and N_2

Can we systematize it?

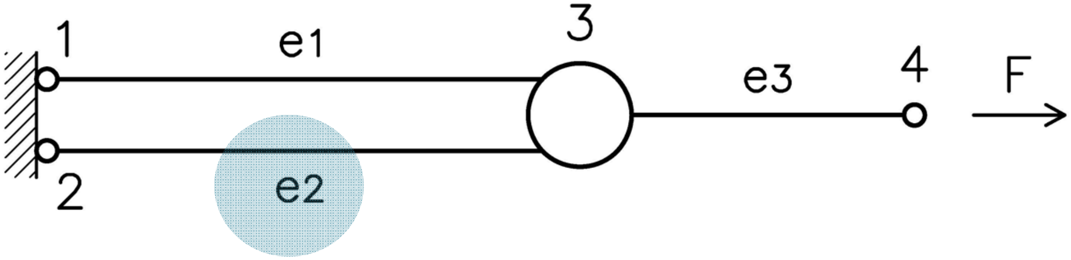
Bar connectivities:



$$\begin{bmatrix} N_1 \\ N_2 \\ N_3 \\ N_4 \end{bmatrix} = \begin{bmatrix} +k_1 & & -k_1 & \\ & +k_2 & -k_2 & \\ -k_1 & -k_2 & +k_1 + k_2 + k_3 & -k_3 \\ & & -k_3 & +k_3 \end{bmatrix} \begin{bmatrix} U_1 \\ U_2 \\ U_3 \\ U_4 \end{bmatrix}$$

Can we systematize it?

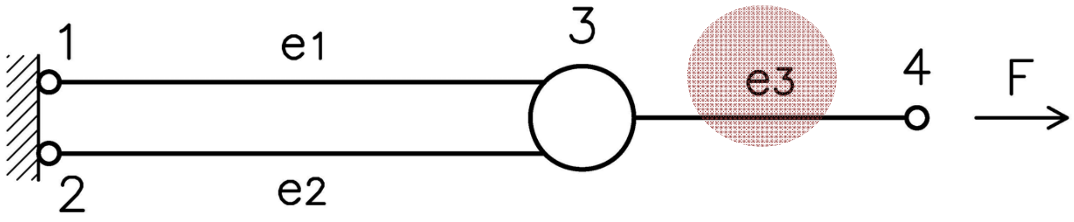
Bar connectivities:



$$\begin{bmatrix} N_1 \\ N_2 \\ N_3 \\ N_4 \end{bmatrix} = \begin{bmatrix} +k_1 & & -k_1 & \\ & +k_2 & -k_2 & \\ -k_1 & -k_2 & +k_1 + k_2 + k_3 & -k_3 \\ & & -k_3 & +k_3 \end{bmatrix} \begin{bmatrix} U_1 \\ U_2 \\ U_3 \\ U_4 \end{bmatrix}$$

Can we systematize it?

Bar connectivities:

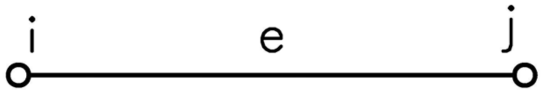


$$\begin{bmatrix} N_1 \\ N_2 \\ N_3 \\ N_4 \end{bmatrix} = \begin{bmatrix} +k_1 & & -k_1 & \\ & +k_2 & -k_2 & \\ -k_1 & -k_2 & +k_1+k_2+k_3 & -k_3 \\ & & -k_3 & +k_3 \end{bmatrix} \begin{bmatrix} U_1 \\ U_2 \\ U_3 \\ U_4 \end{bmatrix}$$

The stiffness matrix is defined adding the stiffness of each bar element at the position of the nodes that define the bar.

Can we systematize it?

In general:

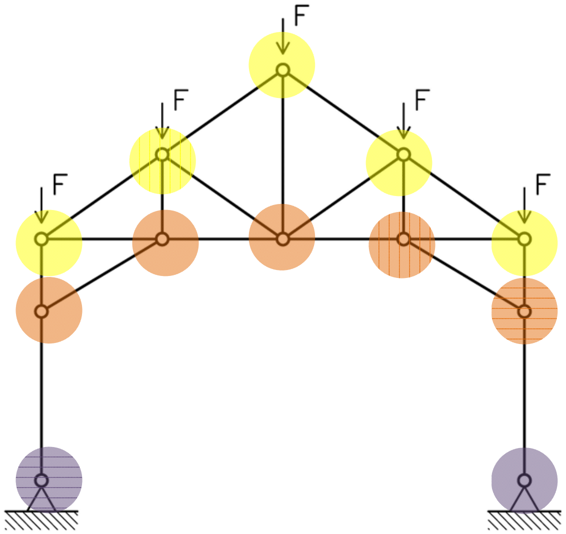


$$\begin{bmatrix}\vdots\\ N_i\\ \vdots\\ N_j\end{bmatrix} = \underset{\text{Column i} \qquad \text{Column j}}{\begin{bmatrix} & \vdots & & \vdots\\ \cdots & +k_e & \cdots & -k_e\\ & \vdots & & \vdots\\ \cdots & -k_e & \cdots & +k_e\end{bmatrix}}\begin{bmatrix}\vdots\\ U_i\\ \vdots\\ U_j\end{bmatrix}\begin{array}{l} \\ \longrightarrow \text{Row i}\\ \\ \longrightarrow \text{Row j}\end{array}$$

The stiffness matrix of the structure is defined by the mechanical characteristics of the bar ke (E, A, L) and the connectivities of the different bar elements.

Can we systematize it?

Boundary Conditions:



🟡 Nodes with known forces (=F)

🟠 Nodes with known forces (=0)

🟣 Nodes with unknown forces but known displacements

To solve the problem we will remove from the system all rows and columns of the 🟣 nodes.

The resulting system of equations is perfectly defined and can be solved to obtain the displacements of all the structure.

Conclusion

Once knowing the displacements of all nodes of the structure it is possible to calculate:

- The reaction forces
- The strains in all bars of the structure
- The stresses of all bars of the structure

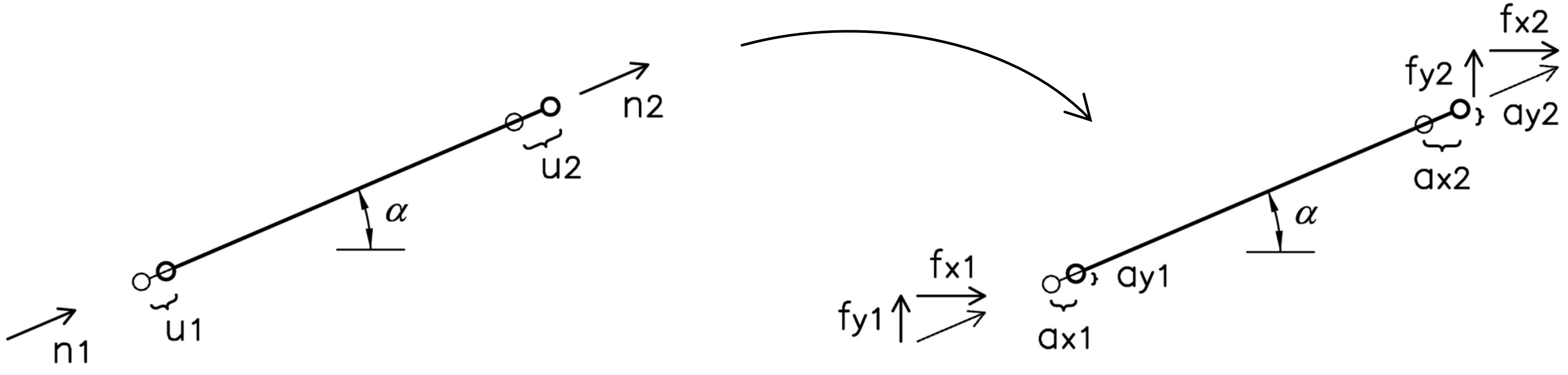
We have found a systematic method to solve linear structures made of truss bars. This can be easily programmed to solve structures as large as wanted.

Possible application:

In which each link is made of a different material!

2D Truss Structures

If the structure is 2D, the bar element can be defined with any possible angle respect the horizontal. In a general case:



where:

$$u_i = a_{xi} \cdot \cos\alpha + a_{yi} \cdot \sin\alpha$$

$$f_{xi} = n_i \cdot \cos\alpha\,; \quad f_{yi} = n_i \cdot \sin\alpha$$

2D Truss Structures

Defining L, the rotation matrix, as: $L = [\cos\alpha \quad \sin\alpha]$

It is possible to rewrite the relations between the un-rotated and the rotated beam as:

$$u_i = [\cos\alpha \quad \sin\alpha] \cdot \begin{bmatrix} a_{xi} \\ a_{yi} \end{bmatrix} = L \cdot \overrightarrow{a_i}$$

$$\overrightarrow{f_i} = \begin{bmatrix} f_{xi} \\ f_{yi} \end{bmatrix} = \begin{bmatrix} \cos\alpha \\ \sin\alpha \end{bmatrix} \cdot n_i = L^T \cdot n_i$$

In the un-rotated configuration:

$$\begin{bmatrix} n_1 \\ n_2 \end{bmatrix} = k \cdot \begin{bmatrix} +1 & -1 \\ -1 & +1 \end{bmatrix} \cdot \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$$

2D Truss Structures

which can be written as:

$$n_1 = +k \cdot u_1 - k \cdot u_2 = +k \cdot L \cdot \overrightarrow{a_1} - k \cdot L \cdot \overrightarrow{a_2}$$

$$n_2 = -k \cdot u_1 + k \cdot u_2 = -k \cdot L \cdot \overrightarrow{a_1} + k \cdot L \cdot \overrightarrow{a_2}$$

and:

$$\overrightarrow{f_1} = L^T \cdot n_1 = L^T \cdot (+k \cdot L \cdot \overrightarrow{a_1} - k \cdot L \cdot \overrightarrow{a_2}) = +k \cdot L^T \cdot L \cdot \overrightarrow{a_1} - k \cdot L^T \cdot L \cdot \overrightarrow{a_2}$$

$$\overrightarrow{f_2} = L^T \cdot n_2 = L^T \cdot (-k \cdot L \cdot \overrightarrow{a_1} + k \cdot L \cdot \overrightarrow{a_2}) = -k \cdot L^T \cdot L \cdot \overrightarrow{a_1} + k \cdot L^T \cdot L \cdot \overrightarrow{a_2}$$

2D Truss Structures

Now it is possible to relate the force and displacement vectors as:

$$\begin{bmatrix} \overrightarrow{f_1} \\ \overrightarrow{f_2} \end{bmatrix} = \begin{bmatrix} f_{x1} \\ f_{y1} \\ f_{x2} \\ f_{y2} \end{bmatrix} = \begin{bmatrix} k_{11} & k_{12} \\ k_{21} & k_{22} \end{bmatrix} \cdot \begin{bmatrix} a_{x1} \\ a_{y1} \\ a_{x2} \\ a_{y2} \end{bmatrix} = \begin{bmatrix} k_{11} & k_{12} \\ k_{21} & k_{22} \end{bmatrix} \cdot \begin{bmatrix} \overrightarrow{a_1} \\ \overrightarrow{a_2} \end{bmatrix}$$

where:

$$k_{11} = +k \cdot L^T \cdot L; \qquad k_{12} = -k \cdot L^T \cdot L$$

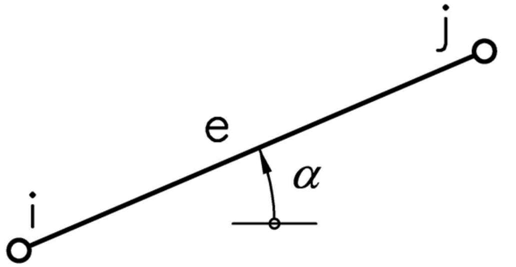
$$k_{21} = -k \cdot L^T \cdot L; \qquad k_{22} = +k \cdot L^T \cdot L$$

being:

$$L^T \cdot L = \begin{bmatrix} \cos\alpha \\ \sin\alpha \end{bmatrix} \cdot \begin{bmatrix} \cos\alpha & \sin\alpha \end{bmatrix} = \begin{bmatrix} \cos\alpha^2 & \cos\alpha \cdot \sin\alpha \\ \cos\alpha \cdot \sin\alpha & \sin\alpha^2 \end{bmatrix}$$

2D Truss Structures

For a general case:



$$\begin{bmatrix}\vdots\\ \vec{f_i}\\ \vdots\\ \vec{f_j}\end{bmatrix} = \begin{bmatrix} & \vdots & & \vdots \\ \cdots & +k_{11}{}^{e} & \cdots & -k_{12}{}^{e} \\ & \vdots & & \vdots \\ \cdots & -k_{21}{}^{e} & \cdots & +k_{22}{}^{e} \end{bmatrix} \begin{bmatrix}\vdots\\ \vec{a_i}\\ \vdots\\ \vec{a_j}\end{bmatrix} \begin{array}{l} \\ \longrightarrow \text{Row i} \\ \\ \longrightarrow \text{Row j} \end{array}$$

→ Column j

→ Column i

with:

$$k_{ij} = (-1)^{i+j}.\frac{E^e \cdot A^e}{L^e} \cdot \begin{bmatrix} \cos\alpha^2 & \cos\alpha\cdot\sin\alpha \\ \cos\alpha\cdot\sin\alpha & \sin\alpha^2 \end{bmatrix}$$

Note: Positive angle counterclockwise

Implementation (1/2)

1. Define the stiffness matrix:
   Size = (2 x # nodes) x (2 x # nodes)

2. For each bar of the structure:
   a. Get/read the bar properties (E, A, L, α)
   b. Compute its stiffness matrix k_{ij}
   c. Add the bar to the structure stiffness matrix in the position i, j defined by the bar nodes (i, j)

3. Define the forces and displacement vectors.
   Size = (2 x # nodes) x 1

4. Fill the force vector with the forces applied to the structure. The rest of the nodes will have force zero or unknown.

Implementation (2/2)

5. Remove the rows and columns of the linear system corresponding to those nodes with the displacement equal to zero (x or y)
6. Solve the linear system of equations

Conclusion

To solve the structure we need to know/define:

1. Know the mechanical properties of the different elements (ki)
2. Know how the elements are connected between them (connectivities)
3. Know which nodes have loads applied
4. Know which nodes have restricted displacements

Any numerical method (matrix solution, FEM, etc.) require the same elements!